

Generalized Einstein Operator Generating Functions

D. CHERNEY^ℳ, E. LATINI^ℒ, AND A. WALDRON^ℳ

^ℳ *Department of Mathematics*
University of California, Davis CA 95616, USA
cherney,wally@math.ucdavis.edu

^ℒ *Department of Mathematics*
University of California, Davis CA 95616, USA
and
INFN, Laboratori Nazionali di Frascati, CP 13, I-00044 Frascati, Italy
emanuele@math.ucdavis.edu, latini@lnf.infn.it

ABSTRACT

We present gauge invariant, self adjoint Einstein operators for mixed symmetry higher spin theories. The result applies to multi-forms, multi-symmetric forms and mixed antisymmetric and symmetric multi-forms, generalizing previous results for combinations of these cases. It also yields explicit action principles for these theories in terms of their minimal covariant field content. For known cases, these actions imply the mixed symmetry equations of motion of Labastida. The result is based on a calculus for handling normal ordered operator expressions built from quantum generators of the underlying constraint algebras.

The dynamics of higher spin fields is described by the equation of motion¹

$$\underbrace{\left(\Delta - Q^i Q_i^* + \frac{1}{2} Q^i Q^j \mathbf{tr}_{ji} \right)}_{\mathbf{G}} \Psi = 0 = \mathbf{tr}_{i(j} \mathbf{tr}_{km]} \Psi, \quad (1)$$

which enjoys the gauge invariance

$$\delta \Psi = Q^k \alpha_k, \quad (2)$$

for gauge parameters α_k subject to the trace condition

$$\mathbf{tr}_{(ij} \alpha_{k]} = 0.$$

¹We refer the reader to the series of higher spin review articles [1].

We refer to \mathbf{G} as the Labastida operator. It acts on mixed multi-forms:

$$\Psi = \Psi_{\mu_1^1 \dots \mu_{s_1}^1 \dots \mu_1^q \dots \mu_{s_q}^q \nu_1^1 \dots \nu_{t_1}^1 \dots \nu_1^r \dots \nu_{t_r}^r} d^1 x^{\mu_1^1} \wedge \dots \wedge d^1 x^{\mu_{s_1}^1} \otimes \dots \otimes d^q x^{\mu_1^q} \wedge \dots \wedge d^q x^{\mu_{s_q}^q} \\ \otimes d^1 x^{\nu_1^1} \odot \dots \odot d^1 x^{\nu_{t_1}^1} \otimes \dots \otimes d^r x^{\nu_1^r} \odot \dots \odot d^r x^{\nu_{t_r}^r},$$

or, in a Young diagrammatic notation

$$\Psi = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \otimes \dots \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \square \square \square \\ \hline \otimes \\ \hline \vdots \\ \hline \otimes \\ \hline \square \square \square \square \end{array}.$$

I.e., an arbitrary representation of $\mathfrak{gl}(d)$ constructed with q columns and r rows. One can think of such tensors as monomials, or better—functions, in the super anti-commuting variables $\eta_i^\mu \equiv d_i x^\mu$ where the superindex

$$i = \underbrace{(1, \dots, q)}_{\text{Fermi}} | \underbrace{(1, \dots, r)}_{\text{Bose}}.$$

The (super)charges (Q_i^*, Q^j) act on blocks of Ψ as exterior derivatives or symmetrized gradients and their duals

$$Q^i = (\mathbf{d}^1, \dots, \mathbf{d}^q | \mathbf{grad}^1, \dots, \mathbf{grad}^r),$$

$$Q_i^* = (\mathbf{d}_1^*, \dots, \mathbf{d}_q^* | \mathbf{div}_1, \dots, \mathbf{div}_r),$$

while Δ is the Laplace operator. The BRST quantization of these operators was first studied in [18]. The algebra of these (super)charges, in an d -dimensional flat spacetime background (for studies of more general backgrounds see [2, 3, 4]),

$$[Q_i^*, Q^j] = \delta_i^j \Delta,$$

may be extended by the R -symmetry generators \mathbf{tr}_{ij}

$$[\mathbf{tr}_{ij}, Q^k] = 2\delta_{[j}^k Q_{i]}.$$

The operator \mathbf{tr}_{ij} traces across the i^{th} and j^{th} blocks of indices while Its dual \mathbf{g}^{ij} uses the metric tensor to add an index to the i^{th} and j^{th} blocks

with supersymmetry between the new indices. These operators generate the full R -symmetry algebras $\mathfrak{osp}(q, q|2r)$ which we use to label the first class algebras $\{Q^i, \Delta, \mathbf{tr}_{ij}, Q_i^*\}$.

In [5] it was shown how BRST detour quantization of these algebras produces the Labastida operators as long operators of detour complexes. The $\mathfrak{sp}(2r)$ version of that computation was given in [6]. Originally, the equation of motion (1) was first given in [7] for the case $r = 0$, while its multi-form analog at $q = 0$ was studied in [8]. The case of mixed multi-forms was first introduced in [9, 10] and studied in detail using BRST detour quantization techniques in [5, 11]. The above equation of motion generalizes the pure multi-form and multi-symmetric form models to the mixed case at arbitrary (q, r) . An unfolded formulation and action principle for mixed higher spins was given in [12] (see also [13] for first order approaches to mixed symmetry fields). The main aim of this Letter is to write action principles for all these theories in terms of a minimal covariant field content.

Einstein Operators

We search for action principles of form

$$S = (\Psi, \mathcal{G}\Psi) \equiv \int \sqrt{-g} \, \Psi^* \mathcal{G}\Psi, \quad (3)$$

which imply the equation of motion (1). Here Ψ^* is defined by replacing oscillator variables by derivatives $(\eta_i^\mu)^* = \frac{\partial}{\partial \eta_i^\mu}$, and the inner product is computed by allowing these to act to the right on the corresponding oscillators in Ψ . We call the operator \mathcal{G} appearing in the action the Einstein operator. It is required to be self-adjoint, in the sense $(\Phi, \mathcal{G}\Psi) = (\mathcal{G}\Phi, \Psi)$.

Another subtlety to bear in mind is that the fields in the Labastida equation of motion obey the double trace constraint $\mathbf{tr}_{i(j} \mathbf{tr}_{km]} \Psi = 0$. Hence, any terms in \mathcal{G} of the form $\mathbf{g}_{i(j} \mathbf{g}_{km]} \mathbf{X}$ will vanish in the inner product. In more mathematical terms, the Einstein operator \mathcal{G} is a mapping

$$\ker(\mathbf{tr}_{i(j} \mathbf{tr}_{km]}) \longrightarrow \text{coker}(\mathbf{g}^{i(j} \mathbf{g}^{km]}).$$

As a warm up we analyze the case of $\mathfrak{sp}(2)$ (*i.e.*, $q = 1, r = 1$), higher spin, totally symmetric tensors

$$\Psi = \Psi_{\mu_1 \dots \mu_t} dx^{\mu_1} \odot \dots \odot dx^{\mu_t}.$$

The Labastida field equation $\mathbf{G}\Psi = 0$ can be brought to a form that comes from the variation of an action by adding a term obtained by tracing the original equation

$$\left(1 - \frac{1}{4}\mathbf{g} \operatorname{tr}\right) \mathbf{G}\Psi \equiv \mathcal{G}\Psi = 0. \quad (4)$$

In this case \mathcal{G} is the desired, manifestly gauge invariant, and self adjoint Einstein operator

$$\mathcal{G} = \Delta - \mathbf{grad} \operatorname{div} + \frac{1}{2} \left(\mathbf{grad}^2 \operatorname{tr} + \mathbf{g} \operatorname{div}^2 \right) - \frac{1}{4} \mathbf{g} \left(2\Delta + \mathbf{grad} \operatorname{div} \right) \operatorname{tr}. \quad (5)$$

When $t = 2$ we may write $\Psi = h_{\mu\nu} dx^\mu \odot dx^\nu$ (*i.e.*, metric fluctuations) and find the linearized Einstein tensor

$$\mathcal{G}\Psi = \left(\Delta h_{\mu\nu} - 2\partial^\rho \partial_\mu h_{\nu\rho} + \partial_\mu \partial_\nu h + \eta_{\mu\nu} \partial^\rho \partial^\sigma h_{\rho\sigma} - \eta_{\mu\nu} \Delta h \right) dx^\mu \odot dx^\nu.$$

This is our motivation for calling \mathcal{G} an “Einstein operator”².

A similar result holds in general but, as the symmetry type of the tensors involved becomes more complicated, higher traces of the Labastida equation of motion must be used to produce an Einstein operator. For the symmetric multi-form case, these trace terms were first computed in [14] (where also the idea for the antisymmetric multi-form case was outlined³.) Earlier results for the symmetric bi-form case were given in [16]. We have found a simple formula for these terms in the most general case of mixed multi-forms based on the BRST detour methodology of [17, 11]. This method has the advantage that it arranges all elements of the BRST cohomology either as field equations, gauge (and gauge for gauge) invariances or Bianchi (and Bianchi for Bianchi) identities. Our result closely mimics the one of Sagnotti *et al* [14] and reads

$$\mathcal{G} = : \frac{I_1(2\sqrt{\omega})}{\sqrt{\omega}} : \mathbf{G}, \quad (6)$$

with

$$\omega = -\frac{\mathbf{g}^{ji} \operatorname{tr}_{ji}}{2}.$$

²One might also be tempted by the term “Maxwell operator”, since when $t = 1$ we have $\Psi = A_\mu dx^\mu$ and equation (5) reproduces Maxwell’s equations. Also, in the $\mathfrak{so}(1, 1)$ case, our Einstein operator again describes Maxwell’s equations and their higher form analogs $\mathbf{d}^* \mathbf{d}\Psi = 0$ (in fact in any curved background, see [11]).

³Those authors have also extended those results to generalized Dirac operators describing fermionic higher spins [15].

Here the symbols $: \bullet :$ indicate normal ordering $(Q^i, \mathbf{g}^{ij}, \Delta, \mathbf{tr}_{ij}, Q_i^*)$ and the modified Bessel function of the first kind divided by the square root of ω has the analytic expansion

$$\frac{I_1(2\sqrt{\omega})}{\sqrt{\omega}} = \sum_n \frac{\omega^n}{n!(n+1)!} \equiv B(\omega). \quad (7)$$

Explicitly, the generating function for the Einstein operator appearing in (3) reads⁴

$$\begin{aligned} \mathcal{G} = & : (B + \omega B') \Delta - Q^i B Q_i^* - Q^k \mathbf{g}^{mj} B' \mathbf{tr}_{mk} Q_j^* \\ & + \frac{1}{2} \mathbf{g}^{ij} B Q_j^* Q_i^* + \frac{1}{2} Q^i Q^j B \mathbf{tr}_{ji} : \end{aligned}$$

Written in the form (6), the gauge invariance (2) is manifest (since identically $\mathbf{G}Q^k \alpha_k = 0$) while the above form makes self-adjointness of \mathcal{G} and, consequently, the Bianchi identity $Q_k^* \mathcal{G} \Psi = 0$ (modulo $\mathbf{g}^{(ij} \mathbf{X}^{k]}$) manifest. However, it is also possible to prove directly the gauge invariance of the action (3) because there is a *calculus* of the above normal ordered operator expressions:

$$: f(\omega) : Q^k = : Q^k f(\omega) + f'(\omega) \mathbf{g}^{ki} Q_i^* :$$

for any function f . This formula converts an apparently difficult algebraic computation into simple differentiations!

Finally, we note that arbitrary symmetry higher spin gauge theory presented here can also be viewed as a detour complex⁵

$$\begin{array}{ccccccc} \cdots \longrightarrow & \ker \mathbf{tr}_{(ij} & \xrightarrow{\mathbf{Q}^k} & \ker \mathbf{tr}_{i(j} \mathbf{tr}_{km]} & \xrightarrow{\text{coker } \mathbf{g}^{i(j} \mathbf{g}^{km]}} & \text{coker } \mathbf{g}^{(ij} & \longrightarrow \cdots \\ & & & \uparrow \mathcal{G} & & & \end{array}$$

⁴ In order to compare with the result proposed in [17] note that $B + \omega B' = I_0(2\sqrt{\omega})$ Bessel function of the first kind.

⁵ We point out that in [9] and [10] a de Rham complex for multi-form higher spin curvatures have been constructed. Our detour complex generalizes that result to mixed multi-forms and is formulated in terms of *gauge potentials*.

whose physical interpretation is the following: $\ker \mathbf{tr}_{i(j)\mathbf{tr}_{km}}$ and $\ker \mathbf{tr}_{(ij)} \equiv \{\alpha_k : \mathbf{tr}_{(ij)\alpha_k} = 0\}$ are the spaces of gauge potentials and gauge parameters and comprise the “incoming complex” which encodes gauge and gauge for gauge symmetries (see [5] and [3] for a complete description). The duals of these spaces, $\text{coker } \mathbf{g}^{(j)}\mathbf{g}^{km}$ and $\text{coker } \mathbf{g}^{(ij)}$, correspond to equations of motion and Noether/Bianchi identities and comprise the “outgoing” complex. The Einstein operator is “long operator” joining the two.

In the following sections we discuss some simple examples.

$\mathfrak{sp}(4)$: Symmetric Multi-Forms

The $q = 0$, $r = 2$ case describes multi-symmetric-form gauge potentials

$$\Psi = \Psi_{\mu_1 \dots \mu_{t_1} \mu_1^2 \dots \mu_{t_2}^2} d^1 x^{\mu_1^1} \odot \dots \odot d^1 x^{\mu_{t_1}^1} \otimes d^2 x^{\mu_1^2} \odot \dots \odot d^2 x^{\mu_{t_2}^2} = \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} . \quad (8)$$

The Bessel series (7), when acting on $\ker \mathbf{tr}_{i(j)\mathbf{tr}_{km}}$, terminates at second order (by virtue of the paucity of index values) so that the generalized Einstein equation of motion is⁶

$$\begin{aligned} \mathcal{G}\Psi &= \left(\Delta - \mathbf{grad}^i \mathbf{div}_i + \frac{1}{2} \mathbf{grad}^i \mathbf{grad}^j \mathbf{tr}_{ji} + \frac{1}{2} \mathbf{g}^{ij} \mathbf{div}_j \mathbf{div}_i - \frac{1}{2} \mathbf{g}^{ij} \Delta \mathbf{tr}_{ji} \right. \\ &\quad + \frac{1}{4} \mathbf{grad}^i \mathbf{g}^{km} \mathbf{tr}_{mk} \mathbf{div}_i - \frac{1}{2} \mathbf{grad}^k \mathbf{g}^{mj} \mathbf{tr}_{mk} \mathbf{div}_j \\ &\quad - \frac{1}{48} \mathbf{grad}^i \mathbf{g}^{km} \mathbf{g}^{pq} \mathbf{tr}_{qp} \mathbf{tr}_{mk} \mathbf{div}_i + \frac{1}{12} \mathbf{grad}^k \mathbf{g}^{mj} \mathbf{g}^{pq} \mathbf{tr}_{qp} \mathbf{tr}_{mk} \mathbf{div}_j \\ &\quad - \frac{1}{6} \mathbf{g}^{i[j} \mathbf{g}^{k]m} \mathbf{tr}_{mk} \mathbf{div}_j \mathbf{div}_i - \frac{1}{6} \mathbf{grad}^i \mathbf{grad}^j \mathbf{g}^{km} \mathbf{tr}_{m[k} \mathbf{tr}_{j]i} \\ &\quad \left. + \frac{1}{16} \mathbf{g}^{km} \mathbf{g}^{pq} \Delta \mathbf{tr}_{qp} \mathbf{tr}_{mk} \right) \Psi = 0 = \mathbf{tr}_{i(j) \mathbf{tr}_{kl}} \Psi . \end{aligned} \quad (9)$$

To construct irreps of $\mathfrak{gl}(d)$ starting from the representation in (8), we introduce the operator \mathbf{N}_1^2 which moves a box from the 1st row to the 2nd row, and note that the Lie algebra

$$\mathfrak{g} = \left\{ \mathbf{grad}^i, \Delta, \mathbf{N}_1^2, \mathbf{tr}_{ij}, \mathbf{div}_i \right\} ,$$

⁶Repairing a factor 4 typographical error in the last term relative to the result of [11].

is first class since

$$[\mathbf{N}_1^2, \mathbf{grad}^1] = \mathbf{grad}^2, \quad \left[\begin{pmatrix} \mathbf{tr}_{12} \\ \mathbf{tr}_{22} \end{pmatrix}, \mathbf{N}_1^2 \right] = \begin{pmatrix} \mathbf{tr}_{11} \\ 2\mathbf{tr}_{12} \end{pmatrix}, \quad [\mathbf{div}_2, \mathbf{N}_1^2] = \mathbf{div}_1.$$

In particular, because \mathbf{N}_1^2 commutes with \mathcal{G} , gauging this operator produces a Dirac constraint $\mathbf{N}_1^2 \Psi = 0$ which selects from the tensor product (8) an irreducible $\mathfrak{gl}(d)$ representation:

$$\mathbf{N}_1^2 \Psi = 0 \quad \rightarrow \quad \Psi = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}. \quad (10)$$

$\mathfrak{osp}(1, 1|2)$: Mixed Multi-Forms

Our final example is the theory of mixed multi-forms

$$\Psi = \Psi_{\mu_1 \dots \mu_{r_1} \nu_1 \dots \nu_{s_1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{r_1}} \otimes dx^{\nu_1} \odot \dots \odot dx^{\nu_{s_1}} = \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}. \quad (11)$$

Explicitly we find, in an obvious notation, the following equation of motion

$$\begin{aligned} \mathcal{G}\Psi &= \left(\Delta - \mathbf{g}^{\text{fb}} \Delta \mathbf{tr}_{\text{bf}} - \frac{1}{2} \mathbf{g}^{\text{bb}} \Delta \mathbf{tr}_{\text{bb}} + \frac{1}{4} \mathbf{g}^{\text{fb}} \mathbf{g}^{\text{fb}} \Delta \mathbf{tr}_{\text{bf}} \mathbf{tr}_{\text{bf}} \right. \\ &\quad - \mathbf{d} \left(1 - \frac{1}{4} \mathbf{g}^{\text{bb}} \mathbf{tr}_{\text{bb}} - \frac{1}{12} \mathbf{g}^{\text{fb}} \mathbf{g}^{\text{fb}} \mathbf{tr}_{\text{bf}} \mathbf{tr}_{\text{bf}} \right) \mathbf{d}^* \\ &\quad - \mathbf{grad} \left(1 + \frac{1}{4} \mathbf{g}^{\text{bb}} \mathbf{tr}_{\text{bb}} - \frac{1}{12} \mathbf{g}^{\text{fb}} \mathbf{g}^{\text{fb}} \mathbf{tr}_{\text{bf}} \mathbf{tr}_{\text{bf}} \right) \mathbf{div} \\ &\quad - \frac{1}{2} \mathbf{d} \mathbf{g}^{\text{bb}} \mathbf{tr}_{\text{bf}} \mathbf{div} - \frac{1}{2} \mathbf{grad} \mathbf{g}^{\text{fb}} \mathbf{tr}_{\text{bb}} \mathbf{d}^* \\ &\quad + \mathbf{g}^{\text{fb}} \left(1 - \frac{1}{2} \mathbf{g}^{\text{fb}} \mathbf{tr}_{\text{bf}} \right) \mathbf{div} \mathbf{d}^* + \mathbf{d} \mathbf{grad} \left(1 - \frac{1}{2} \mathbf{g}^{\text{fb}} \mathbf{tr}_{\text{bf}} \right) \mathbf{tr}_{\text{bf}} \\ &\quad \left. + \frac{1}{2} \mathbf{g}^{\text{bb}} \mathbf{div}^2 + \frac{1}{2} \mathbf{grad}^2 \mathbf{tr}_{\text{bb}} \right) \Psi = 0 = \mathbf{tr}_{\text{bb}}^2 \Psi = \mathbf{tr}_{\text{bb}} \mathbf{tr}_{\text{bf}} \Psi. \end{aligned}$$

Also in this case the irreps of $\mathfrak{gl}(d)$ can be obtained gauging the operator \mathbf{N}_f^b which selects Young tableaux of the shape

$$\mathbf{N}_f^b \Psi = 0 \quad \rightarrow \quad \Psi = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}.$$

Conclusions and Outlook

In this Letter we presented gauge invariant action principles for mixed multi-forms in terms of generating functions for generalized Einstein operators. This gives a simple formulation of mixed symmetry higher spins along with a calculus for handling the operator expressions involved. The results were achieved using the BRST detour quantization techniques of [5], applied to the $\mathfrak{osp}(q, q|2r)$ quantum mechanical models of [2]. Moreover we discussed two interesting examples in which we showed how to construct $\mathfrak{gl}(d)$ irreducible representations. These ideas are easily extended to general cases: irreducible $\mathfrak{gl}(d)$ representations are obtained by studying all possible permutation symmetries, to this end one can introduce operators $\mathbf{N}_i^{j>i}$ which move a box from row j to row i . Since these operators commute with our Einstein operators, their vanishing may be imposed as an additional set of Dirac constraints $\mathbf{N}_i^{j>i}\Psi = 0$ which select particular Young tableaux shapes (a more detailed analysis of this construction may be found in [12, 6]).

Throughout this Letter we worked on flat backgrounds, but the constraint algebra studied can also be represented by tensor operators on manifolds with more interesting geometric structures. Our results can therefore be also applied to those cases. In particular, the $\mathfrak{so}(2, 2)$ case have been analyzed on Kähler manifolds in [17] (Kähler spinning particle models were studied in [4, 19]). The extension of the Einstein generating functions to curved space is also an interesting task; in conformally flat backgrounds the $\mathfrak{osp}(q, q|2r)$ (super)algebras have quadratic corrections built from the Casimirs of their R -symmetry generators so, after minimal covariantization, one could try to construct curved space correction by requiring gauge invariance and adding non-minimal couplings⁷. A BRST detour quantization of that higher order algebra may also yield results for those cases (for BRST studies of (A)dS higher spins see [21]).

Moreover, motivated by the observation that four-dimensional, supersymmetric, black hole solutions to $\mathcal{N} = 2$ supergravities lead to spinning particles with $\mathcal{N} = 4$ worldline supersymmetry [22], we can also analyze those systems. In this case the BRST quantization involves first class constraint algebras with structure functions related to the underlying quaternionic Kähler structures of the spinning particle target spaces; we have found a geometric construction of the BRST charges of these models and will present their quan-

⁷A representation theoretic analysis of this problem has been given in [20].

tization, as well as a gauge invariant action principle, in a future work [23].

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